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CONVEX APPROXIMATION OF INTEGRALS

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Convex approximation of integrals

by

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ABSTRACT

For continuous f , the integral $\int_a^b f(x) dx$ is canonically approximated by the trapezoidal sums

$$T_n(f; a, b) = \frac{1}{n} \left\{ -\frac{1}{2} f(a) + \sum_{k=0}^{n-1} f(a + k(b-a)/n) - \frac{1}{2} f(b) \right\}.$$

In this paper we establish some criteria for these sums to be convex (in n).

KEY WORDS & PHRASES: *convexity, approximations.*

0. INTRODUCTION

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. We define the n -th canonical trapezoidal approximation $T_n(f; a, b)$ of $\int_a^b f(x) dx$ by

$$T_n(f; a, b) = \frac{1}{n} \left\{ -\frac{1}{2} f(a) + \sum_{k=0}^{n-1} f(a + k(b-a)/n) - \frac{1}{2} f(b) \right\}.$$

In this paper we investigate the sums $T_n(x^s; a, b)$ for $s \in \mathbb{R}$. The first named author showed in [2] that the sequence $\{T_n(x^s; 0, 1)\}_{n=1}^{\infty}$ is decreasing for any fixed $s > 1$. This is equivalent to the inequality

$$\sum_{k=1}^n k^s > \frac{1}{2} \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \quad (s > 1).$$

In the first part of this paper we show that for fixed $m \in \mathbb{N}$ the sequence $\{T_n(x^m; 0, 1)\}_{n=1}^{\infty}$ is convex, i.e.

$$2T_n(x^m; 0, 1) \leq T_{n-1}(x^m; 0, 1) + T_{n+1}(x^m; 0, 1).$$

This immediately implies that the sequence $\{T_n(f; 0, b)\}_{n=1}^{\infty}$ is convex if the Taylor expansion of f around the origin converges in $[0, b]$ and has non-negative coefficients. The convexity of the sequence $\{T_n(x^m; 0, 1)\}_{n=1}^{\infty}$ is proved by defining a suitable function $\phi(y)$ such that

$$\phi(n) = T_n(x^m; 0, 1)$$

and checking that $\phi''(y) > 0$ for $y > 0$, so that ϕ is convex.

In the second part of this paper we prove that for fixed $s < 0$ the sequence $\{T_n(x^s; a, b)\}_{n=1}^{\infty}$ is logarithmically convex, i.e.

$$T_n^2(x^s; a, b) \leq T_{n-1}(x^s; a, b) T_{n+1}(x^s; a, b), \quad (0 < a < b; s < 0).$$

The essential step of this prove lies in establishing the convexity of the function

$$\log \left(\frac{1}{x} \frac{e^{\frac{1}{x}} + 1}{\frac{1}{e^x} - 1} \right) \quad \text{for } x > 0,$$

which implies the log-convexity of $\{T_n(e^{\lambda x}; a, b)\}_{n=1}^{\infty}$ for all $\lambda \in \mathbb{R}$, $a < b$.

1. CONVEX APPROXIMATION OF $\int_0^1 x^m dx$, ($m \in \mathbb{N}$).

1.1. Preliminaries; statement of the Theorem

Let $f: [0, 1] \rightarrow \mathbb{R}$ be twice differentiable with continuous second derivative. Then we have by the Euler-Maclaurin summation formula

$$\begin{aligned} T_n(f) &\stackrel{\text{def}}{=} T_n(f; 0, 1) = \frac{1}{n} \left\{ -\frac{1}{2} f(0) + \sum_{k=0}^n f\left(\frac{k}{n}\right) - \frac{1}{2} f(1) \right\} = \\ &= \int_0^1 f(x) dx + \frac{1}{n} \int_0^n \left(x - [x] - \frac{1}{2}\right) df\left(\frac{x}{n}\right). \end{aligned}$$

Let the function $\theta(t)$ be defined by

$$(1) \quad \theta(t) = - \int_0^t \left(x - [x] - \frac{1}{2}\right) dx, \quad t \in \mathbb{R}.$$

Since $\theta(t) = 0$ for $t \in \mathbb{Z}$ we can write

$$\begin{aligned} T_n(f) &= \int_0^1 f(x) dx - \frac{1}{n^2} \int_0^n f'\left(\frac{x}{n}\right) d\theta(x) = \\ &= \int_0^1 f(x) dx + \frac{1}{n^3} \int_0^n f''\left(\frac{x}{n}\right) \theta(x) dx \end{aligned}$$

Now define

$$\phi_f(t) = \frac{1}{t^3} \int_0^t f''\left(\frac{x}{t}\right) \theta(x) dx, \quad t > 0.$$

If f is four times differentiable and if $f''(1) = f^{(3)}(1) = 0$, then $\phi_f(t)$ has a continuous second derivative for $t > 0$, satisfying

$$\phi_f''(t) = \frac{1}{t^4} \int_0^1 (12f''(u) + 8uf^{(3)}(u) + u^2 f^{(4)}(u)) \theta(tu) du.$$

Let $m \in \mathbb{N}$, $m \geq 5$ and put

$$g_m(x) = (1-x)^{m-1}.$$

Note that, by symmetry, $T_n(m) \stackrel{\text{def}}{=} T_n(x^{m-1}) = T_n(g_m(x))$, so that

$$T_n(m) = \frac{1}{m} + \frac{1}{n^3} \int_0^n g_m''\left(\frac{x}{n}\right) \theta(x) dx.$$

Since $g_m''(1) = g_m^{(3)}(1) = 0$, the corresponding function $\phi_m(t) \stackrel{\text{def}}{=} \phi_{g_m}(t)$ satisfies

$$(2) \quad \frac{t^4 \phi_m''(t)}{(m-1)(m-2)} = \int_0^1 \{(m^2 + m)u^2 - 8mu + 12\} (1-u)^{m-5} \theta(tu) du.$$

We intend to prove

THEOREM 1. For every $m \in \mathbb{N}$, the sequence $\{T_n(x^{m-1}; 0, 1)\}_{n=1}^\infty$ is convex.

We shall prove this theorem by showing that the right-hand side of (2), and thus $\phi_m''(t)$, is positive for $m \geq 9$ and $t > 0$. Since by Taylor's theorem

$$\phi_m(n+1) + \phi_m(n-1) = 2\phi_m(n) + \frac{1}{2}(\phi_m''(t_1) + \phi_m''(t_2)),$$

where $t_1 \in (n-1, n)$ and $t_2 \in (n, n+1)$, this implies Theorem 1 for $m \geq 9$. For $m = 1, \dots, 8$ we express $T_n(m)$ by means of the Bernoulli polynomials (Compare for example [1]):

$$T_n(m) = \frac{1}{m} \sum_{0 \leq k < \frac{1}{2}n} \binom{m}{2k} B_{2k} n^{-2k}.$$

For $m = 1, \dots, 8$, the theorem can be verified directly by this formula. So it is sufficient to show that for $t > 0$ and $m \geq 9$

$$(3) \quad \int_0^1 \{(m^2 + m)u^2 - 8mu + 12\}(1-u)^{m-5} \theta(tu) du > 0.$$

1.2. Some Lemma's

LEMMA 1. Let $\theta(t)$ be defined by (1). Then

- a) θ is periodic with period 1.
- b) $\theta(t) = \frac{1}{2}t(1-t)$ for $0 \leq t < 1$.
- c) $\theta(t) \leq \frac{1}{8}$ for all $t \in \mathbb{R}$; $\theta(t) \leq \frac{1}{16}t$ for $t \geq 2$.
- d) $\int_0^n (\theta(t) - \frac{1}{12}) dt = 0$ for $n \in \mathbb{Z}$.
- e) $\int_0^x (\frac{1}{12} - \theta(t)) dt \leq \frac{\sqrt{3}}{216} < \frac{1}{120}$.

PROOF. By straightforward verification from (1). \square

LEMMA 2. If $0 \leq a \leq \frac{1}{2}$ and $0 \leq t \leq 2$, then $\theta(at) \geq \frac{1}{2}t\theta(2a)$.

PROOF. Since $0 \leq at \leq 2a \leq 1$, we have $\theta(at) = \frac{1}{2}at(1-at)$ and $\theta(2a) = a(1-2a)$. Since $0 \leq t \leq 2$, we then have

$$\theta(at) = \frac{1}{2}at(1-at) \geq \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a). \quad \square$$

LEMMA 3. If $0 \leq a \leq \frac{1}{3}$ and $2 \leq t \leq 6$, then $\theta(at) \leq \frac{1}{2}t\theta(2a)$.

PROOF. If $at < 1$ we have by $t \geq 2$

$$\theta(at) = \frac{1}{2}at(1-at) \leq \frac{1}{2}at(1-2a) = \frac{1}{2}t\theta(2a).$$

If $at \geq 1$, then since $1 \leq at \leq 2$ and $0 \leq 2a < 1$

$$\begin{aligned} t\theta(2a) - 2\theta(at) &= at(1-2a) - (at-1)(2-at) \\ &= (at)^2 - 2(1+a)at + 2 \\ &\geq (at)^2 - 2(1+a)at + (1+a)^2 \geq 0. \quad \square \end{aligned}$$

LEMMA 4. If $a \geq \frac{1}{4}$ and $0 \leq x \leq 2$, then

$$\chi_a(x) \stackrel{\text{def}}{=} \int_0^x \theta(at) dt \geq x^2/32.$$

PROOF. Suppose $0 \leq ax \leq 1$. Then we have

$$\begin{aligned} \chi_a(x) &= \int_0^x \theta(at) dt = \frac{1}{a} \int_0^{ax} \theta(u) du = \frac{1}{a} \int_0^{ax} \frac{1}{2} u(1-u) du = \\ &= \frac{1}{12} x (3ax - 2(ax)^2) = \frac{1}{12} x^2 a (3 - 2ax) \geq \frac{1}{12} ax^2. \end{aligned}$$

So the lemma holds if $a \geq 3/8$. But if $a < 3/8$ we have that $ax \leq 6/8$, so $(3 - 2ax) \geq 3/2$. Hence

$$\int_0^x \theta(at) dt = \frac{1}{12} ax^2 (3 - 2ax) \geq \frac{1}{48} x^2 \frac{3}{2} = \frac{1}{32} x^2.$$

Suppose that $ax > 1$. Then by lemma 1(d) and 1(e)

$$\begin{aligned} \int_0^x \theta(at) dt &= \frac{x}{12} + \frac{1}{a} \int_0^{ax} (\theta(u) - \frac{1}{12}) du = \frac{x}{12} - \frac{1}{a} \int_{[ax]}^{ax} (\frac{1}{12} - \theta(u)) du \\ &\geq \frac{x}{12} - \frac{1}{120a} \geq x(\frac{1}{12} - \frac{1}{120}) \geq x^2/32, \end{aligned}$$

since $x \leq 2$. \square

LEMMA 5. For $m \geq 9$ we have

$$I(m) \stackrel{\text{def}}{=} \int_0^6 t(t-2)(t-6)(1-t/m)^{m-5} dt > 0.$$

PROOF. Integration by parts reveals that

$$\begin{aligned} I(m) &= \frac{m^2}{(m-3)(m-4)} \left\{ 12 - 24(1 - \frac{6}{m})^{m-3} - \frac{16m}{m-2} - \frac{20m}{m-2} (1 - \frac{6}{m})^{m-2} + \right. \\ &\quad \left. + \frac{6m^2}{(m-1)(m-2)} - \frac{6m^2}{(m-1)(m-2)} (1 - \frac{6}{m})^{m-1} \right\}. \end{aligned}$$

By direct calculation one may verify that $I(m) > 0$ for $m = 9, \dots, 20$. Since $(1 - 6/m)^m$ increases to its limit e^{-6} we have

$$I(m) > \frac{m^2}{(m-3)(m-4)} \left\{ 2 - \frac{14}{m-1} - \frac{8}{(m-1)(m-2)} + \right. \\ \left. - \left(\frac{24m^3}{(m-6)^3} + \frac{20m^3}{(m-2)(m-6)^2} + \frac{6m^3}{(m-1)(m-2)(m-6)} \right) e^{-6} \right\}.$$

Since the form in curly brackets $\{ \}$ is monotonically increasing in m and is positive for $m = 21$, the proof is complete.

1.3. Proof of the Theorem

Put $h_m(t) \stackrel{\text{def}}{=} (t-2)(t-6)(1-t/m)^{m-5}$. We shall prove that for $a > 0$ and $m \geq 9$

$$(4) \quad \int_0^6 h_m(t) \theta(at) dt > 0.$$

Since $h_m(t) \geq 0$ for $t \geq 6$ and since $\theta(t) \geq 0$ for all t , (4) implies

$$\int_0^m h_m(t) \theta(at) dt > 0,$$

so that, putting $u = t/m$ and $y = am$,

$$\int_0^1 (m^2 y^2 - 8mu + 12)(1-u)^{m-5} \theta(uy) du > 0,$$

which implies (3) and the Theorem. Hence it is sufficient to show (4). Now suppose that $0 < a < \frac{1}{4}$. By Lemmas 2 and 3 we have

$$\int_0^6 h_m(t) \theta(at) dt \geq \left\{ \int_0^2 + \int_2^6 \right\} h_m(t) \frac{1}{2} t \theta(2a) dt = \\ = \frac{1}{2} \theta(2a) \int_0^6 t h_m(t) dt > 0,$$

by Lemma 5.

So let $a \geq \frac{1}{4}$ and as before, put $\chi_a(x) = \int_0^x \theta(at)dt$. Since $\chi_a(0) = 0 = h_m(2)$, we have that

$$I_1(m) \stackrel{\text{def}}{=} \int_0^2 h_m(t) \theta(at) dt = \int_0^2 h_m(t) d\chi_a(t) = - \int_0^2 \chi_a(t) d h_m(t).$$

Observe that $h_m(t)$ is decreasing for $0 \leq t \leq 2$. We thus have by Lemma 4

$$I_1(m) \geq - \int_0^2 \frac{t^2}{32} dh_m(t) = \int_0^2 h_m(t) \frac{t}{16} dt.$$

Since $h_m(t) \leq 0$ for $2 \leq t \leq 6$ we have by Lemma 1

$$I_2(m) \stackrel{\text{def}}{=} \int_2^6 h_m(t) \theta(at) dt \geq \int_2^6 h_m(t) \frac{t}{16} dt.$$

Hence, by Lemma 5

$$\int_0^6 h_m(t) \theta(at) dt = I_1(m) + I_2(m) \geq \frac{1}{16} \int_0^6 t h_m(t) dt > 0$$

for $m \geq 9$, completing the proof of Theorem 1.

1.4. An inequality for $T_n(m)$; conclusion

Theorem 1 reads

$$T_{n-1}(m) + T_{n+1}(m) \geq 2 T_n(m), \quad (m, n \in \mathbb{N}; n \geq 2).$$

Since

$$(n+1)^m T_{n+1}(m) = n^m T_n(m) + \frac{1}{2} n^{m-1} + \frac{1}{2} (n+1)^{m-1},$$

we can write the above inequality as

$$T_n(m) \left(\left(\frac{n}{n-1} \right)^m - 2 + \left(\frac{n}{n+1} \right)^m \right) \geq \frac{1}{2} \frac{n^{m-1}}{(n-1)^m} - \frac{1}{2} \frac{n^{m-1}}{(n+1)^m} + \frac{1}{n^2 - 1},$$

or

$$T_n(m) \geq \frac{1}{2} \frac{n^{m-1}(n+1)^m + 2(n+1)^{m-1}(n-1)^{m-1} + n^{m-1}(n-1)^m}{(n+1)^m n^m - 2(n+1)^m (n-1)^m + n^m (n-1)^m}.$$

The method in this section can be applied to functions x^s with $s \geq 9$. For $s < 9$ and $s \notin \mathbb{N}$ we cannot prove anything. However, numerical evidence supports the following stronger conjecture.

Conjecture. For any fixed real $s > 1$ the sequence $\{T_n(x^s; 0, 1)\}_{n=1}^\infty$ is logarithmically convex.

2. LOGARITHMICALLY CONVEX APPROXIMATION OF $\int_{\alpha}^{\beta} x^{-s} dx$, $s > 0$.

2.1. Preliminaries

A sequence $\{a_n\}_{n=1}^\infty$ is called logarithmically convex (or log-convex) if $a_n \geq 0$ for all $n \in \mathbb{N}$ and if $a_n^2 \leq a_{n-1} a_{n+1}$ for all $n \geq 2$. If $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are log-convex then $\{pa_n\}_{n=1}^\infty$, ($p > 0$), $\{a_n b_n\}_{n=1}^\infty$ and $\{a_n + b_n\}_{n=1}^\infty$ are log-convex. The first two results are trivial, the last one is proved by means of the Cauchy-Schwartz inequality. Moreover we have

LEMMA 2.1. Let $\{A_n(t)\}_{n=1}^\infty$ be a log-convex sequence for each $t \in [\alpha, \beta]$. If $p(t) \geq 0$, then the sequence $\{b_n\}_{n=1}^\infty$, given by

$$b_n = \int_{\alpha}^{\beta} p(t) A_n(t) dt, \quad (n = 1, 2, \dots)$$

is log-convex.

PROOF. Write $a_n(t) = \sqrt{A_n(t)}$. We have

$$\begin{aligned} b_n^2 &= \left(\int_{\alpha}^{\beta} p(t) A_n(t) dt \right)^2 \leq \left(\int_{\alpha}^{\beta} p(t) a_{n-1}(t) a_{n+1}(t) dt \right)^2 \leq \\ &\leq \int_{\alpha}^{\beta} p(t) a_{n-1}^2(t) dt \int_{\alpha}^{\beta} p(t) a_{n+1}^2(t) dt = b_{n-1} b_{n+1}. \quad \square \end{aligned}$$

2.2. Convexity of $\{T_n(e^{-\lambda x}; \alpha, \beta)\}_{n=1}^{\infty}$

The following lemma is essential.

LEMMA 2.3. *The function*

$$K(x) = \frac{1}{x} \frac{e^{\frac{1}{x}} + 1}{\frac{1}{e^{\frac{1}{x}} - 1}}, \quad x \in \mathbb{R}^+$$

satisfies

$$(\log K(x))'' \geq 0.$$

PROOF. Define $\phi(x) = \log K(x)$. Observe that

$$\phi''(x) = \frac{1}{x^2} - \frac{4}{x^3 (e^{\frac{1}{x}} - e^{-\frac{1}{x}})} + \frac{2(e^{\frac{1}{x}} + e^{-\frac{1}{x}})}{x^4 (e^{\frac{1}{x}} - e^{-\frac{1}{x}})^2}.$$

Setting $u = \frac{1}{x}$ we need to show that for $u > 0$

$$1 - \frac{4u}{e^u - e^{-u}} + \frac{2u^2(e^u + e^{-u})}{(e^u - e^{-u})^2} > 0$$

or, equivalently, that

$$(6) \quad e^{4u} - 2e^{2u} + 1 - 4u(e^{3u} - e^u) + 2u^2(e^{3u} + e^u) > 0.$$

The left-hand side is an entire function of u with power series expansion

$$\sum_{n=0}^{\infty} c_n u^n,$$

say.

Now observe that $c_0 = c_1 = 0$ and that for $n \geq 2$

$$c_n = \frac{1}{n!} (4^n - 2^{n+1} - 4n3^{n-1} + 4n + 2n(n-1)3^{n-2} + 2n(n-1)).$$

Hence $c_2 = 0$, $c_3 = 0$, $c_4 = 2$, $c_5 = 4$, $c_6 = 77/18$. For $n \geq 7$ we have

$$n!c_n > -4n3^{n-1} + 2n(n-1)3^{n-2} = 2n(n-7)3^{n-2} \geq 0,$$

so that $c_n \geq 0$ for $n = 0, 1, 2, \dots$. This proves (6) and the lemma. \square

We now prove

THEOREM 2. Let $\lambda \in \mathbb{R}$ be fixed and let $(a, b) \subset \mathbb{R}$. Then the sequence

$$\{T_n(e^{\lambda x}; a, b)\}_{n=1}^{\infty}$$

is logarithmically convex (in n).

PROOF. Put $\Delta = b - a$. We have

$$\begin{aligned} T_n(e^{\lambda x}; a, b) &= \frac{1}{2n} \sum_{k=0}^{n-1} \{e^{\lambda(a+k\Delta/n)} + e^{\lambda(a+(k+1)\Delta/n)}\} = \\ &= \frac{1}{2} e^{\lambda a} \frac{e^{\lambda\Delta/n} - 1}{\lambda\Delta/n} \frac{\lambda\Delta/n}{e^{\lambda\Delta/n} - 1}. \end{aligned}$$

Since $\frac{1}{2} e^{\lambda a} (e^{\lambda\Delta/n} - 1)/\lambda\Delta$ is positive, we must show that the sequence

$$\{K(\frac{n}{\lambda\Delta})\}_{n=1}^{\infty}$$

is log-convex. For $\lambda > 0$ this follows from Lemma 2.3. For $\lambda < 0$ observe that $K(x) = K(-x)$. For $\lambda = 0$ the theorem is trivial. \square

2.2. The main Theorem

THEOREM 3. Let $s > 0$ be fixed and let $b > a > 0$. Then the sequence

$$\{T_n(x^{-s}; a, b)\}_{n=1}^{\infty}$$

is logarithmically convex.

PROOF. For $s > 0$ and $x > 0$ we have

$$\Gamma(s) = \int_0^{\infty} e^{-u} u^{s-1} du = x^s \int_0^{\infty} e^{-xt} t^{s-1} dt.$$

so that

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt} t^{s-1} dt.$$

Since T_n acts as a linear operator, we have for $0 < a < b$

$$T_n(x^{-s}; a, b) = \frac{1}{\Gamma(s)} \int_0^{\infty} T_n(e^{-xt}; a, b) t^{s-1} dt.$$

Since each sequence $T_n(e^{-xt}; a, b)$ is log-convex, the theorem follows directly from Lemma 2.1. \square

Theorem 2.1 can be generalized as follows. Let $\{c_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that

$$f(x) = \sum_{k=1}^{\infty} c_k x^{-k}$$

is convergent for $x \in [a, b]$. Then

$$\begin{aligned} T_n(f; a, b) &= \sum_{k=1}^{\infty} c_k T_n(x^{-k}; a, b) = \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(k)} \left(\int_0^{\infty} T_n(e^{-xt}; a, b) t^{k-1} dt \right) \\ &= \int_0^{\infty} g(t) T_n(e^{-xt}; a, b) dt, \end{aligned}$$

where $g(t) = \sum_{k=1}^{\infty} \frac{c_k}{\Gamma(k)} t^{k-1}$. If $g(t)$ converges for $t \in \mathbb{R}^+$ and is non-negative on \mathbb{R}^+ , then it follows that $T_n(f)$ is log-convex.

EXAMPLE. Let $f(x) = \sum_{k=1}^{2m+1} (-1)^{k+1} x^{-k}$. Then $g(t) = 1 - \frac{t}{1!} + \frac{t^2}{2!} - \dots + \frac{t^{2m}}{2m!}$.

Since $e^{-t} = g(t) - \frac{t^{2m+1}}{(2m+1)!} e^{-\eta t}$ for some $\eta \in (0,1)$ by Taylor's theorem, we find that $g(t) > 0$ for all $t \in \mathbb{R}^+$. So $\{T_n(f; \alpha, \beta)\}_{n=1}^\infty$ is log-convex. The above argument can be directly extended to functions of the form

$$f(x) = \sum_{k=1}^{\infty} c_k x^{-s_k},$$

where $0 < s_1 < s_2 < \dots$ are real numbers, the c_k 's satisfying similar conditions as above. The reader will have no difficulties in constructing an integral analogue of the above generalization of theorem 2.1.

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